

Comment on "Essential discreteness in generalized thermostatistics with non-logarithmic entropy" by S. Abe

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Abstract

Recently Abe (arXiv:cond-mat/1005.5110v1) claimed that the q -entropy of nonextensive statistical mechanics cannot be generalized for the continuous variables and therefore can be used *only* in the discrete case. In this letter, we show that the discrete q -entropy can be generalized to continuous variables *exactly in the same manner as* Boltzmann-Gibbs entropy, contrary to the claim by Abe, so that q -entropy can be used with discrete as well as continuous variables.

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1 Introduction

As Abe very recently reminded [1], Boltzmann-Gibbs (BG) entropy has originated mainly for the case when the random variable takes finite number of values [2]. When this is the case, BG entropy, setting Boltzmann constant to unity, is given by

$$S(p_i) = - \sum_{i=1}^n p_i \ln p_i. \quad (1)$$

In order to generalize the discrete expression above to the continuous case, the usual practice is to assume the existence of a probability density function $p(x)$, which is equal to or greater than zero in some interval $[a, b]$. In addition to this, the criterion of normalization is assumed to be satisfied by $p(x)$ in the interval $[a, b]$. Then, in a straightforward manner, Eq. (1) is extended to the continuous variables as

$$S(p) = - \int_a^b p(x) \ln p(x) dx. \quad (2)$$

Although this generalization by itself seems reasonable, three serious drawbacks can be noted at once. First, probability densities usually have a unit of inverse of length in one dimensional continuum. Then, $S(p)$ possesses an overall unit of $\log(\text{length})$ in contrast to the dimensionless discrete entropy expression $S(p_i)$ in Eq. (1). Second, $S(p)$ is not invariant with respect to coordinate transformations deeming it vulnerable to Bertrand paradox [2]. The last but not the least, the discrete BG entropy $S(p_i)$ in the $n \rightarrow \infty$ limit and $S(p)$ yield different results. In order to show this, we consider a uniform distribution in the interval $[a, b]$ i.e.,

$$p(x) = \frac{1}{b-a} \quad (3)$$

such that the corresponding probabilities in the discrete case, assuming that the interval $[a, b]$ is divided in n equal subintervals, are of the form

$$p(x_i) = \frac{1}{n} \quad (4)$$

where $i = 1, 2, \dots, n$. The calculation of the continuous entropy $S(p)$ with the probability density given by Eq. (3) results in

$$S(p) = \ln(b-a) \quad (5)$$

whereas the discrete form $S(p_i)$ together with Eq. (4) yield

$$S(p_i) = \ln(n). \quad (6)$$

The discrete entropy $S(p_i)$ attains the infinity in the $n \rightarrow \infty$ limit, whereas the continuous version $S(p)$ is independent of n , and again equal to $\ln(b-a)$. Summing up, the so-called continuous version $S(p)$ of the discrete BG entropy $S(p_i)$ is not tenable and cannot be used in the continuum.

Although the solution to this dilemma is known in the case of BG entropy in detail [2] as reviewed in the next section too for the sake of completeness of our treatment, Abe [1] recently argued that the same cannot be said for the q -entropy proposed by Tsallis [3, 4] and deduced that the q -entropy is properly functional only in the discrete case. In this paper, we show that the q -entropy can indeed be generalized to the continuous case as naturally as BG entropy so that Abe's desideratum is satisfied and therefore the conclusion reached by Abe is shown to be unfounded.

2 Extension to the Continuum: Boltzmann-Gibbs entropy

In this section we review the generalization of BG entropy in the case of random variables, since q -entropy is generalized almost in the same manner as BG entropy. Despite the fact that one can have different derivations of this generalization depending on the level of rigor, we follow Abe's treatment [1], which is identical to the one by Jaynes (see p. 375 in Ref. [5] for example). In order to extend BG entropy to the continuum, we consider an interval $[a, b]$ so that the discrete points x_i with $i = 1, 2, \dots, n$ fill the interval. Then, the relation between the discrete probability p_i and the probability density $\rho(x_i)$ is given by [1, 5]

$$p_i \rightarrow \frac{\rho(x_i)}{nm(x_i)}. \quad (7)$$

As the number of points increases and tends to infinity, one can write [1, 5]

$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{1}{nm(x_i)} = \int_a^b dx. \quad (8)$$

Substitution of Eq. (7) into the discrete entropy expression given by Eq. (1) yields

$$S = - \sum_{i=1}^n \frac{\rho(x_i)}{nm(x_i)} \ln \left(\frac{\rho(x_i)}{nm(x_i)} \right). \quad (9)$$

Eq. (9) can now be rewritten as

$$S = - \sum_{i=1}^n \frac{\rho(x_i)}{nm(x_i)} \ln \left(\frac{\rho(x_i)}{m(x_i)} \right) + \sum_{i=1}^n \frac{\rho(x_i)}{nm(x_i)} \ln(n) \quad (10)$$

so that Eq. (10), using Eq. (8) in the $n \rightarrow \infty$ limit, becomes

$$S = - \int_a^b dx \rho(x) \ln \left(\frac{\rho(x)}{m(x)} \right) + \ln(n) \int_a^b dx \rho(x). \quad (11)$$

It is crucial to understand that the left hand side of the equation above is discrete BG entropy given by Eq. (1), but when one considers it in the $n \rightarrow \infty$ limit i.e., continuous limit. Eq. (11), due to normalization, can be rewritten as

$$S = - \int_a^b dx \rho(x) \ln \left(\frac{\rho(x)}{m(x)} \right) \quad (12)$$

where the additive divergent term $\lim_{n \rightarrow \infty} \ln(n)$ is omitted since the entropy is not absolute but only its change can be measured. Then, this equation indicates that the continuous form of BG entropy (remember that the left hand side of the equation above is discrete BG entropy but in the $n \rightarrow \infty$ limit) is given by the negative of the relative entropy i.e., right hand side of Eq. (12) as discussed in [1, 5]. The integral in the right hand side i.e., the extension of BG entropy to continuum is called relative entropy or Kullback-Leibler entropy, since it was first proposed by Kullback and Leibler [6]. It is worth remarking that the relative entropy expression given by Eq. (12) is dimensionless like its discrete counterpart, and invariant under different reparametrization of continuum.

3 Extension to the Continuum: q -entropy

BG entropy is the cornerstone of statistical mechanics and its maximization yields exponential distributions. On the other hand, there are many physical systems exhibiting inverse power law distributions (see Ref. [4] for a survey). Therefore, the discrete q -entropy was proposed [3, 4] as a generalization of BG entropy measure in order to investigate certain classes of such systems. The nonadditive q -entropy reads

$$S_q(p_i) = \sum_{i=1}^n p_i \ln_q \left(\frac{1}{p_i} \right) = - \sum_{i=1}^n p_i \ln_{2-q}(p_i) \quad (13)$$

where q -logarithm is defined as

$$\ln_q(x) = \frac{x^{1-q} - 1}{1 - q} \quad (14)$$

for $x > 0$. Note that q -logarithm becomes the ordinary logarithmic function in the $q \rightarrow 1$ limit so that the discrete q -entropy in Eq. (13) takes the form of ordinary BG entropy.

In order to extend the discrete q -entropy definition in Eq. (13) to continuum, we consider an interval $[a, b]$ filled by the discrete points x_i where the index i runs up to n , exactly as in the previous section. Then, the relation between the discrete probability p_i and the probability density $\rho(x_i)$ [1, 5] is given by Eq. (7) with $m \rightarrow m_q$, where $m_q(x_i)$ is the q -generalized $m(x_i)$ measure, relevant for a nonextensive system, as mentioned also in Ref. [1], and present the essential difference between ordinary and q -statistics. Therefore, one can write

$$S_q = - \sum_{i=1}^n \frac{\rho(x_i)}{n m_q(x_i)} \ln_{2-q} \left(\frac{\rho(x_i)}{n m_q(x_i)} \right). \quad (15)$$

Let us now define the $m_q(x_i)$ measure as follows

$$m_q(x_i) := \frac{\rho(x_i)}{n} \left[\left(\frac{\rho(x_i)}{m(x_i)} \right) \oslash_{2-q} n \right]^{-1} \quad (16)$$

where \oslash_{2-q} is the generalized division originally introduced in [7, 8] as $x \oslash_q y = [x^{1-q} - y^{1-q} + 1]^{1/(1-q)}$, for $q \rightarrow 2 - q$ and $m_{q \rightarrow 1}(x_i) = m(x_i)$. The measure $m_q(x_i)$ is consistently defined since it has again the dimension of a probability and let Δx_i tend to zero for $n \rightarrow \infty$. Then, by virtue of Eq. (16), we obtain from Eq. (15)

$$S_q = - \sum_{i=1}^n \frac{\rho(x_i)}{n m_q(x_i)} \ln_{2-q} \left(\frac{\rho(x_i)}{m(x_i)} \right) + \ln_{2-q}(n) \sum_{i=1}^n \frac{\rho(x_i)}{n m_q(x_i)}. \quad (17)$$

Using Eq. (8) in the $n \rightarrow \infty$ limit with $m \rightarrow m_q$, we can write Eq. (17) in the continuous case as

$$S_q = - \int_a^b dx \rho(x) \ln_{2-q} \left(\frac{\rho(x)}{m(x)} \right) + \ln_{2-q}(n) \int_a^b dx \rho(x). \quad (18)$$

It is clearly seen that the left hand side of this equation is discrete q -entropy given by Eq. (13) when it is considered in the $n \rightarrow \infty$ limit. Eq. (18), due to normalization, can be rewritten as

$$S_q = - \int_a^b dx \rho(x) \ln_{2-q} \left(\frac{\rho(x)}{m(x)} \right), \quad (19)$$

since the additive divergent term $\lim_{n \rightarrow \infty} \ln_{2-q}(n)$ is omitted due to the fact that the entropy is not absolute but only its change can be measured. This result can also be obtained by plugging the definition in Eq. (16) into Eq. (10) of Ref. [1]. This last equation i.e., Eq. (18), is essential, since it shows that the continuous form of q -entropy (just like BG case in the previous section, note that the left hand side of the equation above is discrete q -entropy but in the $n \rightarrow \infty$ limit) is given by the negative of the corresponding q -relative entropy i.e., the integral in the right hand side of Eq. (19) [9].

It can now be seen that none of the divergences mentioned by Abe [1] occurs in Eq. (19), since the divergence is, again like BG entropy, additive and therefore can be omitted (compare Eq. (10) in Ref. [1] and Eq. (19) in the present manuscript). Moreover, the relative entropy expression given by Eq. (19) is dimensionless and invariant under different reparametrization of continuum just like its BG counterpart.

4 Conclusions

Contrary to the recent claim by Abe [1], we have shown that the discrete q -entropy can be generalized to the continuous variables exactly in the same manner as done for the BG entropy. Moreover, the resulting generalization of the q -entropy to the continuum is dimensionless and independent of reparametrization of the continuous variables just as the continuous BG entropy.

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